

Construction of Multivariate Gaussian Weyl–Heisenberg Frames, (I)

Maurice de Gosson*
Universität Wien, NuHAG
Fakultät für Mathematik
A-1090 Wien

December 16, 2010

Abstract

(This Note replaces a former note which contains an incorrect proof) .Let ϕ be an arbitrary generalized Gaussian (squeezed coherent state), $\Lambda_{\alpha\beta} = (\alpha_1\mathbb{Z} \times \cdots \times \alpha_n\mathbb{Z}) \times (\beta_1\mathbb{Z} \times \cdots \times \beta_n\mathbb{Z})$ a rectangular lattice. We show that there exists a positive definite symplectic matrix M (depending on ϕ) such that the multivariate Weyl–Heisenberg system $\mathcal{G}(\phi, M\Lambda_{\alpha\beta})$ is a frame. In a forthcoming Note we will prove a converse to this result.

1 Introduction

Let $\phi \in L^2(\mathbb{R}^n)$, $\phi \neq 0$ and a lattice $\Lambda \subset \mathbb{R}^{2n}$. For $z_0 \in \mathbb{R}^{2n}$ define the Heisenberg operator $\widehat{T}(z_0) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ by

$$\widehat{T}(z_0)\psi = e^{\frac{i}{\hbar}(p_0 \cdot x - \frac{1}{2}p_0 \cdot x_0)}\psi(x - x_0) \quad (1)$$

(\hbar is a positive constant, usually taken to be $1/2\pi$ in time-frequency analysis, and to $h/2\pi$ in quantum mechanics; h is Planck’s constant). The set

*Financed by the Austrian Research Agency FWF (Projektnummer P20442-N13).

$\mathcal{G}(\phi, \Lambda) = \{\widehat{T}(z)\phi : z \in \Lambda\}$ is called a Weyl–Heisenberg (or Gabor) system. If $\mathcal{G}(\phi, \Lambda)$ is a frame in $L^2(\mathbb{R}^n)$, i.e. [7] if there exist $a, b > 0$ such that

$$a\|\psi\|^2 \leq \sum_{z \in \Lambda} |(\psi|\widehat{T}(z)\phi)|^2 \leq b\|\psi\|^2 \quad (2)$$

for every $\psi \in L^2(\mathbb{R}^n)$ then $\mathcal{G}(\phi, \Lambda)$ is called a Weyl–Heisenberg (or Gabor) frame.

Remark 1 *The function $z \mapsto (\psi|\widehat{T}(z)\phi)$ is, up to the factor $(2\pi\hbar)^n$, the cross-ambiguity function $A(\psi, \phi) = F_\sigma W(\psi, \phi)$ (F_σ the symplectic Fourier transform and $W(\psi, \phi)$ the cross-Wigner distribution).*

A particularly interesting situation occurs when one chooses a Gaussian window ϕ because Gaussians play a privileged role in both time-frequency analysis and quantum mechanics [3, 4, 5, 7]. A classical result is the following necessary and sufficient condition in the case $n = 1$, due to Lyubarski [11] and Seip and Wallstén [13]:

Proposition 2 *Let $\phi_1(x) = (\pi\hbar)^{-1/4}e^{-x^2/2\hbar}$ (the “fiducial coherent state”) with $x \in \mathbb{R}$ and $\Lambda_{\alpha\beta} = \alpha\mathbb{Z} \times \beta\mathbb{Z}$. The Gabor system $\mathcal{G}(\phi_1, \Lambda_{\alpha\beta})$ is a frame for $L^2(\mathbb{R})$ if and only if $\alpha\beta < 2\pi\hbar$.*

This result has the following non-trivial extension, proven in [1, 2]:

Proposition 3 *Let $\phi = \phi_1 \otimes \cdots \otimes \phi_1$ and $\Lambda_{\alpha\beta} = (\alpha_1\mathbb{Z} \times \cdots \times \alpha_n\mathbb{Z}) \times (\beta_1\mathbb{Z} \times \cdots \times \beta_n\mathbb{Z})$. Then $\mathcal{G}(\phi, \Lambda_{\alpha\beta})$ is a frame if and only if $\alpha_j\beta_j < 2\pi\hbar$ for $1 \leq j \leq n$.*

The problem of constructing multivariate Weyl–Heisenberg systems $\mathcal{G}(\phi, \Lambda)$ with an arbitrary Gaussian ϕ and lattice Λ is reputedly difficult and has been tackled by many authors (see the comments in [9], and the review in [8]). That problem however becomes more easily tractable if one recasts it in terms of phase-space objects such that Heisenberg operators and cross-Wigner function, which allows one to use the full power of the symplectic covariance machinery familiar to mathematical physicists working in phase space quantum mechanics [4, 5]. We are going to show that:

Proposition 4 *Let the lattice $\Lambda_{\alpha\beta} = \alpha\mathbb{Z}^n \times \beta\mathbb{Z}^n$ be defined as above; let $\phi_{X,Y}$ be a “squeezed coherent state”, that is a Gaussian of the type*

$$\phi_{X,Y}(x) = \left(\frac{1}{\pi\hbar}\right)^{n/4} (\det X)^{1/4} e^{-\frac{1}{2\hbar}(X+iY)x^2} \quad (3)$$

where $X + iY$ is a complex symmetric $n \times n$ matrix with real part $X > 0$. Let $G = S^T S$ be the positive definite symplectic matrix where

$$S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix}. \quad (4)$$

The Weyl–Heisenberg system $\mathcal{G}(\phi_{X,Y}, G^{-1/2}\Lambda_{\alpha\beta})$ is a frame if and only if $\alpha_j\beta_j < 2\pi\hbar$ for $1 \leq j \leq n$.

2 Two Lemmas

Let $\text{Mp}(2n, \mathbb{R})$ be the metaplectic group; we denote by $\pi^{\text{Mp}} : \text{Mp}(2n, \mathbb{R}) \rightarrow \text{Sp}(2n, \mathbb{R})$ the natural projection onto the symplectic group. Recall that $\text{Mp}(2n, \mathbb{R})$ is a double cover of $\text{Sp}(2n, \mathbb{R})$ consisting of unitary operators on $L^2(\mathbb{R}^n)$ [3, 4, 5]. We recall the following covariance property of the Heisenberg operator:

$$\widehat{S}\widehat{T}(z) = \widehat{T}(Sz)\widehat{S}, \quad S = \pi^{\text{Mp}}(\widehat{S}). \quad (5)$$

Lemma 5 *Let $\mathcal{G}(\phi, \Lambda)$ be a Weyl–Heisenberg system, and $\widehat{S} \in \text{Mp}(2n, \mathbb{R})$, $S = \pi^{\text{Mp}}(\widehat{S})$. Then $\mathcal{G}(\phi, \Lambda)$ is a frame in $L^2(\mathbb{R}^n)$ if and only if $\mathcal{G}(\widehat{S}\phi, S\Lambda)$ is a frame in $L^2(\mathbb{R}^n)$.*

Proof. (See [5], Chapter 8). We have, using (5),

$$\sum_{z \in S\Lambda} |(\psi|\widehat{T}(z)\widehat{S}\phi)|^2 = \sum_{z \in S\Lambda} |(\psi|\widehat{S}\widehat{T}(S^{-1}z)\phi)|^2 = \sum_{z \in \Lambda} |(\widehat{S}^{-1}\psi|\widehat{T}(z)\phi)|^2$$

hence the result since $\|\widehat{S}^{-1}\psi\| = \|\psi\|$. ■

The second Lemma gives an explicit formula for the Wigner transform of a squeezed coherent state. Recall that for $\phi \in L^2(\mathbb{R}^n)$

$$W\psi(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar}p \cdot y} \psi\left(x + \frac{1}{2}y\right) \overline{\psi\left(x - \frac{1}{2}y\right)} dy. \quad (6)$$

Lemma 6 *Let $\phi_{X,Y}$ be the Gaussian (3). We have*

$$W\phi_{X,Y}(z) = \left(\frac{1}{\pi\hbar}\right)^n e^{-\frac{1}{\hbar}Gz^2} \quad (7)$$

where $G \in \text{Sp}(2n, \mathbb{R})$ is the positive-definite matrix

$$G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix} = S^T S \quad (8)$$

where the symplectic matrix S is given by (4).

Proof. See [4, 5]. ■

3 Proof of Proposition 4

Recall [4, 5] the following symplectic covariance property of the Wigner transform:

$$W(\psi, \phi)(S^{-1}z) = W(\widehat{S}\psi, \widehat{S}\phi)(z). \quad (9)$$

We note that $\mathcal{G}(\phi, \Lambda)$ is a frame if and only if $\mathcal{G}(c\phi, \Lambda)$ is a frame when $c \in \mathbb{C}$ is a complex number.

The matrix G defined by (8) in Lemma 6 is both positive-definite and symplectic hence there exists $U \in \text{Sp}(2n, \mathbb{R}) \cap O(2n, \mathbb{R})$ such that $UGU^T = D$ is diagonal [3, 4]. Let $\lambda_1, \dots, \lambda_{2n}$ be the eigenvalues of G . Since the eigenvalues of a positive-definite symplectic matrix occur in pairs $(\lambda_j, 1/\lambda_j)$ we may assume that $\lambda_1 \geq \dots \geq \lambda_n \geq 1$, hence these eigenvalues λ_j can be ordered as follows: $\lambda_1 \geq \dots \geq \lambda_n \geq 1 \geq \lambda_n^{-1} \geq \dots \geq \lambda_1^{-1}$. We thus have

$$G = U^T D U = U^T \begin{pmatrix} \Delta & 0 \\ 0 & \Delta^{-1} \end{pmatrix} U$$

with

$$\Delta = \text{diag}(\lambda_1, \dots, \lambda_n) \quad (10)$$

and hence

$$G^{-1/2} = U^T \begin{pmatrix} \Delta^{-1/2} & 0 \\ 0 & \Delta^{1/2} \end{pmatrix} U \in \text{Sp}(2n, \mathbb{R}).$$

We are going to show that the Gaussian $\phi_{X,Y}$ becomes a tensor product of elementary one-dimensional Gaussians if transformed by a suitable metaplectic operator. Let in fact $\widehat{S}_G \in \text{Mp}(2n, \mathbb{R})$ be one of the two metaplectic operators such that $\pi^{\text{Mp}}(\widehat{S}_G) = G^{-1/2}$. Formulas (7) and (9) imply that

$$W(\widehat{S}_G \phi_{X,Y})(z) = W(\phi_{X,Y})(G^{-1/2}z) = \left(\frac{1}{\pi\hbar}\right)^n e^{-\frac{1}{\hbar}|z|^2} \quad (11)$$

and hence $\widehat{S}_G \phi_{X,Y} = c \phi_1 \otimes \cdots \otimes \phi_1$ for some complex constant with $|c| = 1$. Thus, in view of Lemma 5, $\mathcal{G}(\phi_{X,Y}, G^{-1/2} \Lambda_{\alpha\beta})$ is a frame if and only if $\mathcal{G}(\phi_1 \otimes \cdots \otimes \phi_1, \Lambda_{\alpha\beta})$ is a frame. But this is the case if and only if $\alpha_j \beta_j < 2\pi\hbar$ for $1 \leq j \leq n$ in view of Proposition 3.

References

- [1] A. Bourouihiya, Beurling Weighted Spaces, Product-Convolution Operators, and the Tensor Product of Frames, Doctoral Dissertation, University of Maryland, College Park USA, directed by Professor John J. Benedetto (2006).
- [2] A. Bourouihiya, The tensor product of frames, *Sampl. Theory Signal Image Process.* **7**(1) (2008), 65–76.
- [3] G. B. Folland, *Harmonic Analysis in Phase space*, Annals of Mathematics studies, Princeton University Press, Princeton, N.J. (1989)
- [4] M. de Gosson, *Symplectic Geometry and Quantum Mechanics*, Birkhäuser, Basel, series “Operator Theory: Advances and Applications” (subseries: “Advances in Partial Differential Equations”), Vol. **166** (2006).
- [5] M. de Gosson, *Symplectic Methods in Harmonic Analysis; Applications to Mathematical Physics*, Birkhäuser, 2011 (in press).
- [6] M. de Gosson and F. Luef, *Symplectic Capacities and the Geometry of Uncertainty: the Irruption of Symplectic Topology in Classical and Quantum Mechanics*, *Physics Reports*, **484** (2009), 131–179 179DOI 10.1016/j.physrep.2009.08.001.
- [7] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, (2000).
- [8] K. Gröchenig, Multivariate Gabor Frames and Sampling of Entire Functions of Several Variables, Preprint (November 2010).
- [9] K. Gröchenig and Yu. Lyubarskii, *Gabor (super)frames with Hermite functions*, *Math. Annal.* **345** (2009), 267–286.

- [10] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math., **82** (1985), 307–347.
- [11] Yu. I. Lyubarskii, *Frames in the Bargmann space of entire functions*, In Entire and subharmonic functions, Amer. Math. Soc., Providence RI (1992), 167–180.
- [12] L. Polterovich, *The Geometry of the Group of Symplectic Diffeomorphisms*, Lectures in Mathematics, Birkhäuser, (2001).
- [13] K. Seip and R. Wallstén, *Density theorems for sampling and interpolation in the Bargmann–Fock space. II*, J. Reine Angew. Math **429** (1992), 107–113.
- [14] J. Williamson, *On the algebraic problem concerning the normal forms of linear dynamical systems*, Amer. J. of Math. 58 (1936) 141–163.

E-Mail: maurice.de.gosson@univie.ac.at